

# DYCK PATHS AND POSITROIDS FROM UNIT INTERVAL ORDERS

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**ABSTRACT.** It is well known that the number of non-isomorphic unit interval orders on  $[n]$  equals the  $n$ -th Catalan number. Using work of Skandera and Reed and work of Postnikov, we show that each unit interval order on  $[n]$  naturally induces a rank  $n$  positroid on  $[2n]$ . We call the positroids produced in this fashion *unit interval positroids*. We characterize the unit interval positroids by describing their associated decorated permutations, showing that each one must be a  $2n$ -cycle encoding a Dyck path of length  $2n$ .

## 1. INTRODUCTION

A *unit interval order* is a partially ordered set that captures the order relations among a collection of unit intervals on the real line. Unit interval orders were introduced by Luce [9] to axiomatize a class of utilities in the theory of preferences in economics. Since then they have been systematically studied (see [4, 5, 6, 7, 15] and references therein). These posets exhibit many interesting properties; for example, they can be characterized as the posets that are simultaneously  $(\mathbf{3} + \mathbf{1})$ -free and  $(\mathbf{2} + \mathbf{2})$ -free. Moreover, it is well known that the number of non-isomorphic unit interval orders on  $[n]$  equals  $\frac{1}{n+1} \binom{2n}{n}$ , the  $n$ -th Catalan number (see [4, Section 4] or [16, Exercise 2.180]).

In [15], motivated by the desire to understand the  $f$ -vectors of various classes of posets, Skandera and Reed showed that one can canonically label the elements of a unit interval order from 1 to  $n$  so that its  $n \times n$  antiadjacency matrix has all its minors nonnegative<sup>1</sup> and its zero entries form a right-justified Young diagram located strictly above the main diagonal and anchored in the upper-right corner. The zero entries of such a matrix are separated from the one entries by a Dyck path joining the upper-left corner to the lower-right corner. Motivated by this observation, we call such matrices *Dyck matrices*. The Hasse diagram and the antiadjacency (Dyck) matrix of a canonically labeled unit interval order are shown in Figure 1.

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<sup>1</sup>We avoid the term *totally nonnegative* in the context of square matrices and instead reserve it for rectangular matrices whose maximal minors are all nonnegative.

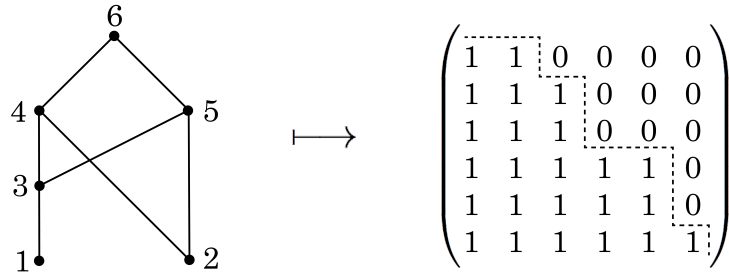


FIGURE 1. A canonically 6-labeled unit interval order and its antiadjacency matrix, which exhibits its *semiorder path*, i.e., the Dyck path separating its one entries from its zero entries.

On the other hand, it follows from work of Postnikov [11] that  $n \times n$  Dyck matrices can be regarded as representing rank  $n$  *positroids* on the ground set  $[2n]$ . A positroid is a matroid that can be represented by a full-rank totally nonnegative  $d \times n$  matrix. Positroids were introduced and classified by Postnikov in his study of the totally nonnegative part of the Grassmannian [11]. He showed that positroids are in bijection with various interesting families of combinatorial objects, including decorated permutations and Grassmann necklaces. Positroids and the nonnegative Grassmannian have been the subject of a great deal of recent work, with connections and applications to cluster algebras [14], scattering amplitudes [3], soliton solutions to the KP equation [8], and free probability [2].

In this paper we characterize the positroids that arise from unit interval orders, which we call *unit interval positroids*. We show that the decorated permutations of rank  $n$  unit interval positroids are certain  $2n$ -cycles in bijection with Dyck paths of length  $2n$ . The following theorem is a formal statement of our main result.

**Main Theorem.** *A decorated permutation  $\pi$  represents a unit interval positroid on  $[2n]$  if and only if  $\pi$  is a  $2n$ -cycle  $(1 \ j_1 \ \dots \ j_{2n-1})$  satisfying the following two conditions:*

- (1) *in the sequence  $(1, j_1, \dots, j_{2n-1})$  the elements  $1, \dots, n$  appear in increasing order while the elements  $n+1, \dots, 2n$  appear in decreasing order;*
- (2) *for every  $1 \leq k \leq 2n-1$ , the set  $\{1, j_1, \dots, j_k\}$  contains at least as many elements of the set  $\{1, \dots, n\}$  as elements of the set  $\{n+1, \dots, 2n\}$ .*

*In particular, there are  $\frac{1}{n+1} \binom{2n}{n}$  unit interval positroids on  $[2n]$ .*

The decorated permutation associated to a unit interval positroid on  $[2n]$  naturally encodes a Dyck path of length  $2n$ . Here we provide a recipe to read such decorated permutation directly from the antiadjacency matrix of the unit interval order.

**Theorem 1.1.** *Let  $P$  be a canonically  $n$ -labeled unit interval order and  $A$  the antiadjacency matrix of  $P$ . If we number the  $n$  vertical steps of the semiorder (Dyck) path of  $A$  from bottom to top in increasing order with  $\{1, \dots, n\}$  and the  $n$  horizontal steps*

from left to right in increasing order with  $\{n+1, \dots, 2n\}$ , then we obtain the decorated permutation of the unit interval positroid induced by  $P$  by reading the semiorder (Dyck) path in northwest direction.

**Example 1.2.** The vertical assignment on the left of Figure 2 shows a set  $\mathcal{I}$  of unit intervals along with a canonically 5-labeled unit interval order  $P_5$  describing the order relations among the intervals in  $\mathcal{I}$  (see Theorem 2.2). The vertical assignment on the right illustrates the recipe given in Theorem 1.1 to read the decorated permutation  $\pi = (1 \ 2 \ 10 \ 3 \ 9 \ 4 \ 8 \ 7 \ 5 \ 6)$  associated to the unit interval positroid induced by  $P_5$  directly from the antiadjacency matrix. Note that the decorated permutation  $\pi$  is a 10-cycle satisfying conditions (1) and (2) of our main theorem. The solid and dashed assignment signs represent functions that we shall introduce later.

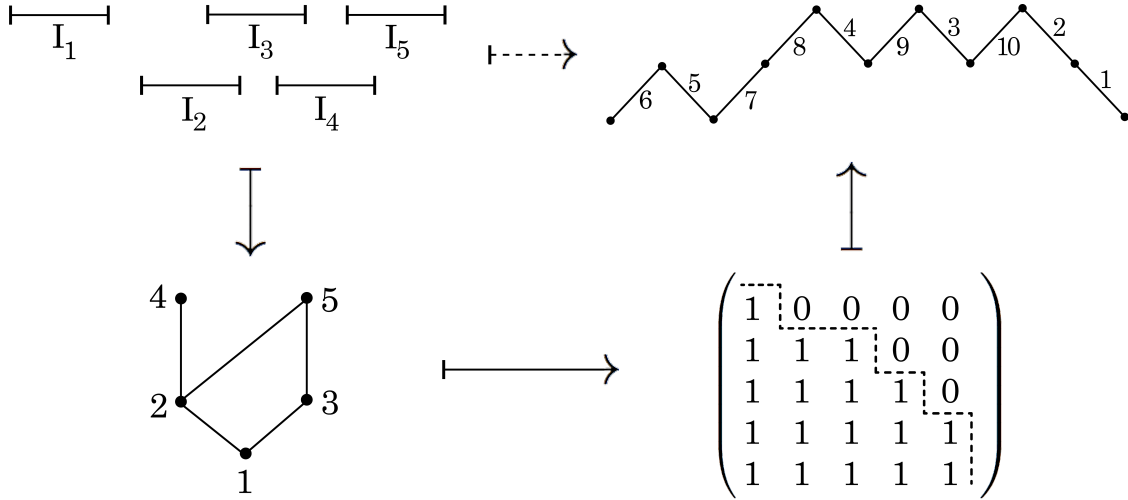


FIGURE 2. Following the solid assignments: unit interval representation  $\mathcal{I}$ , its unit interval order  $P_5$ , the antiadjacency matrix  $\varphi(P_5)$ , and the semiorder (Dyck) path of  $\varphi(P_5)$  showing the decorated permutation  $\pi$ .

This paper is organized as follows. In Section 2 we establish the notation and formally present the fundamental concepts and objects used throughout this sequel. Then, in Section 3, we formally introduce canonical labelings and canonical interval representations of unit interval orders. Also, we use canonical labelings to exhibit an explicit bijection from the set of non-isomorphic unit interval orders of cardinality  $n$  to the set of  $n \times n$  Dyck matrices. Section 4 is dedicated to the description of the unit interval positroids via their decorated permutations, which yields the direct implication of the main theorem. Finally, in Section 5, we show how to read the decorated permutation of a unit interval positroid from either an antiadjacency matrix or a canonical interval representation of the corresponding unit interval order, which allows us to complete the proof of the main theorem.

## 2. BACKGROUND AND NOTATION

For ease of notation, when  $(P, <_P)$  is a partially ordered set (*poset* for short), we just write  $P$ , tacitly assuming that the order relation on  $P$  is to be denoted by the symbol  $<_P$ . In addition, every poset showing up in this paper is assumed to be finite. Let  $P$  and  $Q$  be two posets. We say that  $Q$  is an *induced* subposet of  $P$  if there exists an injective map  $f: Q \rightarrow P$  such that for all  $r, s \in Q$  one has  $r <_Q s$  if and only if  $f(r) <_P f(s)$ . By contrast,  $P$  is a  *$Q$ -free* poset if  $P$  does not contain any induced subposet isomorphic to  $Q$ .

**Definition 2.1.** A *unit interval order* is a poset that is simultaneously  $(\mathbf{3} + \mathbf{1})$ -free and  $(\mathbf{2} + \mathbf{2})$ -free.

For each  $n \in \mathbb{N}$ , we denote by  $\mathcal{U}_n$  the set comprised of all non-isomorphic unit interval orders of cardinality  $n$ . As mentioned in the introduction, unit interval orders exhibit many interesting properties; the following, for instance, is a curious characterization that explains their name.

**Theorem 2.2.** [12, Theorem 2.1] *A poset  $P$  is a unit interval order if and only if there exists an injective map  $i \mapsto [q_i, q_i + 1]$  from  $P$  to the set of closed unit intervals of the real line such that for distinct  $i, j \in P$ ,  $i <_P j$  if and only if  $q_i + 1 < q_j$ .*

If the poset  $P$  has cardinality  $n$ , then a bijective function  $\ell: P \rightarrow [n]$  is called an  *$n$ -labeling* of  $P$ ; after identifying  $P$  with  $[n]$  via  $\ell$ , we say that  $P$  is an  *$n$ -labeled* poset. The  $n$ -labeled poset  $P$  is *naturally labeled* if  $i <_P j$  implies that  $i \leq j$  as integers for all  $i, j \in P$ .

Motivated by Theorem 2.2, we say that a finite collection  $\{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  of unit intervals is an *interval representation* of an  $n$ -labeled unit interval order  $P$  provided that for distinct  $i, j \in P$  we have  $i <_P j$  if and only if  $q_i + 1 < q_j$ . According to Theorem 2.2, every  $n$ -labeled unit interval order has an interval representation.

**Example 2.3.** The figure below illustrates the 6-labeled unit interval order introduced in Figure 1 with a corresponding interval representation.

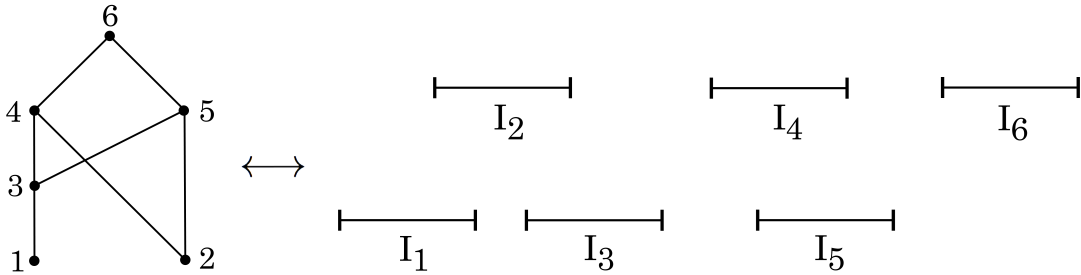


FIGURE 3. A 6-labeled unit interval order and one of its interval representations.

Another useful way of representing  $n$ -labeled unit interval orders is through their *antiadjacency matrix*.

**Definition 2.4.** If  $P$  is an  $n$ -labeled poset, then the *antiadjacency matrix* of  $P$  is the  $n \times n$  binary matrix  $A = (a_{i,j})$  with  $a_{i,j} = 0$  if and only if  $i \neq j$  and  $i <_P j$ .

Recall that a binary square matrix is said to be a *Dyck matrix* if its zero entries form a right-justified Young diagram strictly above the main diagonal and anchored in the upper-right corner. All minors of a Dyck matrix are nonnegative (see, for instance, [1]). We denote by  $\mathcal{D}_n$  the set of all  $n \times n$  Dyck matrices. As presented in [15], every unit interval order can be naturally labeled so that its antiadjacency matrix is a Dyck matrix (details provided in Section 3). This yields a natural map  $\varphi: \mathcal{U}_n \rightarrow \mathcal{D}_n$  that is a bijection (see Theorem 3.5). In particular,  $|\mathcal{D}_n|$  is the  $n$ -th Catalan number, which can also be deduced from the one-to-one correspondence between Dyck matrices and their semiorder (Dyck) paths.

A real matrix is said to be *totally nonnegative* if all its maximal minors are nonnegative. Given an  $n \times n$  matrix  $A$  all whose minors are nonnegative, there is a natural assignment  $A \mapsto \phi(A)$ , where  $\phi(A)$  is a full-rank totally nonnegative  $n \times 2n$  matrix.

**Lemma 2.5.** [11, Lemma 3.9] *For an  $n \times n$  real matrix  $A = (a_{i,j})$ , consider the  $n \times 2n$  matrix  $B = \phi(A)$ , where*

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} 1 & \cdots & 0 & 0 & (-1)^{n-1}a_{n,1} & \cdots & (-1)^{n-1}a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -a_{2,1} & \cdots & -a_{2,n} \\ 0 & \cdots & 0 & 1 & a_{1,1} & \cdots & a_{1,n} \end{pmatrix}.$$

*Under this correspondence,  $\Delta_{I,J}(A) = \Delta_{([n] \setminus I) \cup (n+J)}(B)$  for all  $I, J \subseteq [n]$  satisfying  $|I| = |J|$  (here  $\Delta_{I,J}(A)$  is the minor of  $A$  determined by the rows  $I$  and columns  $J$ , and  $\Delta_K(B)$  is the maximal minor of  $B$  determined by columns  $K$ ).*

Using Lemma 2.5 and the aforementioned map  $\varphi: \mathcal{U}_n \rightarrow \mathcal{D}_n$ , we can assign via  $\phi \circ \varphi$  a full-rank totally nonnegative  $n \times 2n$  real matrix to each unit interval order of cardinality  $n$ . Every full-rank totally nonnegative real matrix gives rise to a positroid, a special representable matroid which has a very rich combinatorial structure. Let us recall the concept of matroid.

**Definition 2.6.** Let  $E$  be a finite set, and let  $\mathcal{B}$  be a nonempty collection of subsets of  $E$ . The pair  $M = (E, \mathcal{B})$  is a *matroid* if for all  $B, B' \in \mathcal{B}$  and  $b \in B \setminus B'$ , there exists  $b' \in B' \setminus B$  such that  $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$ .

If  $M = (E, \mathcal{B})$  is a matroid, then the elements of  $\mathcal{B}$  are said to be *bases* of  $M$ . Any two bases of  $M$  have the same size, which we denote by  $r(M)$  and call *rank* of  $M$ . There is a special subfamily of matroids determined by full-rank totally nonnegative real matrices.

**Definition 2.7.** For  $d, n \in \mathbb{N}$  such that  $d \leq n$ , let  $A$  be a  $d \times n$  totally nonnegative real matrix of rank  $d$  whose columns are denoted by  $A_1, \dots, A_n$ . The subsets  $B$  of  $[n]$  such that  $\{A_b \mid b \in B\}$  is a basis for  $\mathbb{R}^d$  are the bases of a matroid  $M(A)$ . Such a matroid is called *positroid*.

Each unit interval order  $P$  (labeled so that its antiadjacency matrix is a Dyck matrix) induces a positroid via Lemma 2.5, namely, the positroid represented by the full-rank totally nonnegative matrix  $\phi(\varphi(P))$ .

**Definition 2.8.** A positroid on  $[2n]$  induced by a unit interval order is called *unit interval positroid*.

We denote by  $\mathcal{P}_n$  the set of all unit interval positroids on the ground set  $[2n]$ . The function  $\rho \circ \phi \circ \varphi: \mathcal{U}_n \rightarrow \mathcal{P}_n$ , where  $\rho(B)$  is the positroid represented by a full-rank totally nonnegative real matrix  $B$ , plays a fundamental role in this paper. Indeed, we will end up proving that it is a bijection (see Theorem 5.4). However, at this point it might not even be clear why  $\rho \circ \phi \circ \varphi$  is well defined.

Several families of combinatorial objects, in bijection with positroids, were introduced in [11] to study the positive Grassmannian: decorated permutations, Grassmann necklaces, Le-diagrams, and plabic graphs. Although each nontrivial positroid has many matrix representations, some of the mentioned families of combinatorial objects can be used to represent positroids in a concise and unique manner. We use *decorated permutations*, obtained from *Grassmann necklaces*, to provide a compact and elegant description of unit interval positroids.

**Definition 2.9.** Let  $d, n \in \mathbb{N}$  such that  $d \leq n$ . An  $n$ -tuple  $(I_1, \dots, I_n)$  of  $d$ -subsets of  $[n]$  is called a *Grassmann necklace* of type  $(d, n)$  if for every  $i \in [n]$  the following conditions hold:

- $i \in I_i$  implies  $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$  for some  $j \in [n]$ ;
- $i \notin I_i$  implies  $I_{i+1} = I_i$ .

We are assuming that  $I_{n+1} = I_1$ .

For  $i \in [n]$ , the total order  $<_i$  on  $[n]$  defined by  $i <_i \dots <_i n <_i 1 <_i \dots <_i i - 1$  is called *shifted linear  $i$ -order*. For a matroid  $M = ([n], \mathcal{B})$  of rank  $d$ , one can define the sequence  $\mathcal{I}(M) = (I_1, \dots, I_n)$ , where  $I_j$  is the lexicographically minimal ordered basis of  $M$  with respect to the shifted linear  $i$ -order. It was proved in [11, Section 16] that the sequence  $\mathcal{I}(M)$  is a Grassmann necklace of type  $(d, n)$ . When  $M$  is a positroid we can recover  $M$  from its Grassmann necklace (see, e.g., [10] and [11]).

For  $i \in [n]$ , the *Gale order* on  $\binom{[n]}{d}$  with respect to  $<_i$  is the partial order  $\prec_i$  defined in the following way. If  $S = \{s_1 <_i \dots <_i s_d\} \subseteq [n]$  and  $T = \{t_1 <_i \dots <_i t_d\} \subseteq [n]$ , then  $S \prec_i T$  if and only if  $s_j <_i t_j$  for each  $j \in [d]$ .

**Theorem 2.10.** [10, Theorem 6] *For  $d, n \in \mathbb{N}$  such that  $d \leq n$ , let  $\mathcal{I} = (I_1, \dots, I_n)$  be a Grassmann necklace of type  $(d, n)$ . Then*

$$\mathcal{B}(\mathcal{I}) = \left\{ B \in \binom{[n]}{d} \mid I_j \prec_j B \text{ for every } j \in [n] \right\}$$

*is the collection of bases of a positroid  $M(\mathcal{I}) = ([n], \mathcal{B}(\mathcal{I}))$ , where  $\prec_i$  is the Gale  $i$ -order on  $\binom{[n]}{d}$ . Moreover,  $M(\mathcal{I}(M)) = M$  for all positroids  $M$ .*

Therefore there is a natural bijection between positroids on  $[n]$  of rank  $d$  and Grassmann necklaces of type  $(d, n)$ . However, *decorated permutations*, also in one-to-one correspondence with positroids, will provide a more succinct representation.

**Definition 2.11.** A *decorated permutation* of  $[n]$  is an element  $\pi \in S_n$  whose fixed points  $j$  are marked either “clockwise” (denoted by  $\pi(j) = \underline{j}$ ) or “counterclockwise” (denoted by  $\pi(j) = \bar{j}$ ).

A *weak  $i$ -excedance* of a decorated permutation  $\pi \in S_n$  is an index  $j \in [n]$  satisfying  $j <_i \pi(j)$  or  $\pi(j) = \bar{j}$ . It is easy to see that the number of weak  $i$ -excedances does not depend on  $i$ , so we just call it the number of *weak excedances*.

To every Grassmann necklace  $\mathcal{I} = (I_1, \dots, I_n)$  one can associate a decorated permutation  $\pi_{\mathcal{I}}$  as follows:

- if  $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ , then  $\pi_{\mathcal{I}}(j) = i$ ;
- if  $I_{i+1} = I_i$  and  $i \notin I_i$ , then  $\pi_{\mathcal{I}}(i) = \underline{i}$ ;
- if  $I_{i+1} = I_i$  and  $i \in I_i$ , then  $\pi_{\mathcal{I}}(i) = \bar{i}$ .

The assignment  $\mathcal{I} \mapsto \pi_{\mathcal{I}}$  defines a one-to-one correspondence between the set of Grassmann necklaces of type  $(d, n)$  and the set of decorated permutations of  $[n]$  having exactly  $d$  weak excedances.

**Proposition 2.12.** [2, Proposition 4.6] *The map  $\mathcal{I} \mapsto \pi_{\mathcal{I}}$  is a bijection between the set of Grassmann necklaces of type  $(d, n)$  and the set of decorated permutations of  $[n]$  having exactly  $d$  weak excedances.*

### 3. CANONICAL LABELINGS ON UNIT INTERVAL ORDERS

In this section we introduce the concept of *canonically* labeled poset, and we use it to exhibit an explicit bijection from the set  $\mathcal{U}_n$  of non-isomorphic unit interval orders of cardinality  $n$  to the set  $\mathcal{D}_n$  of  $n \times n$  Dyck matrices.

Given a poset  $P$  and  $i \in P$ , we denote the *order ideal* and the *dual order ideal* of  $i$  by  $\Lambda_i$  and  $V_i$ , respectively. The *altitude* of  $P$  is the map  $\alpha: P \rightarrow \mathbb{Z}$  defined by  $i \mapsto |\Lambda_i| - |V_i|$ . An  $n$ -labeled poset  $P$  *respects altitude* if for all  $i, j \in P$ , the fact that  $\alpha(i) < \alpha(j)$  implies  $i < j$  (as integers). Notice that every poset can be labeled by the set  $[n]$  such that, as an  $n$ -labeled poset, it respects altitude.

**Definition 3.1.** An  $n$ -labeled poset is *canonically labeled* if it respects altitude.

Each canonically  $n$ -labeled poset is, in particular, naturally labeled. The next proposition characterizes canonically  $n$ -labeled unit interval orders in terms of their antiadjacency matrices.

**Proposition 3.2.** [15, Proposition 5] *An  $n$ -labeled unit interval order is canonically labeled if and only if its antiadjacency matrix is a Dyck matrix.*

The above proposition indicates that the antiadjacency matrices of canonically labeled unit interval orders are quite special. In addition, canonically labeled unit interval orders have very convenient interval representations.

**Proposition 3.3.** *Let  $P$  be an  $n$ -labeled unit interval order. Then the labeling of  $P$  is canonical if and only if there exists an interval representation  $\{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  of  $P$  such that  $q_1 < \dots < q_n$ .*

*Proof.* Let  $\alpha: P \rightarrow \mathbb{Z}$  be the altitude map of  $P$ . For the forward implication, suppose that the  $n$ -labeling of  $P$  is canonical. The existence of an interval representation of  $P$  is guaranteed by Theorem 2.2. Among all interval representations of  $P$ , let us assume that  $\{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  gives us the maximum  $m \in [n]$  such that  $q_1 < \dots < q_m$ . Suppose, by way of contradiction, that  $m < n$ . The maximality of  $m$  implies that  $q_m > q_{m+1}$ . This, along with the fact that  $\alpha(m) \leq \alpha(m+1)$ , ensures that  $q_m \in (q_{m+1}, q_{m+1} + 1)$ . Similarly,  $q_i + 1 \notin (q_{m+1}, q_m)$  for any  $i \in [n]$ ; otherwise

$$\alpha(m+1) = |\Lambda_{m+1}| - |V_{m+1}| < |\Lambda_m| - |V_{m+1}| \leq |\Lambda_m| - |V_m| = \alpha(m)$$

would contradict that the  $n$ -labeling of  $P$  respects altitude. An analogous argument guarantees that  $q_i \notin (q_{m+1} + 1, q_m + 1)$  for any  $i \in [n]$ .

Now take  $k$  to be the smallest natural in  $[m]$  such that  $q_j > q_{m+1}$  for all  $j \geq k$ , and take  $\sigma = (k \ k+1 \ \dots \ m \ m+1) \in S_n$ . We will show that  $\{[p_i, p_i + 1] \mid 1 \leq i \leq n\}$ , where  $p_i = q_{\sigma(i)}$ , is an interval representation of  $P$ . Take  $i, j \in P$  such that  $i <_P j$ . Since  $i$  and  $j$  are comparable in  $P$ , at least one of them must be fixed by  $\sigma$ ; say  $\sigma(i) = i$ . If  $\sigma(j) = j$ , then  $p_i + 1 = q_i + 1 < q_j = p_j$ . Also, if  $\sigma(j) \neq j$ , then  $q_i + 1 < q_j \in (q_{m+1}, q_m)$ . It follows from  $q_i + 1 < q_m$  that  $p_i + 1 = q_i + 1 < q_{m+1} < q_{\sigma(j)} = p_j$ . The case of  $\sigma(j) = j$  can be argued analogously. Because  $p_i + 1 < p_j$  when  $i <_P j$ , the set  $\{[p_i, p_i + 1] \mid 1 \leq i \leq n\}$  is an interval representation of  $P$ . As  $q_1 < \dots < q_m$ , the definition of  $k$  implies that  $p_1 < \dots < p_{m+1}$ , which contradicts the maximality of  $m$ . Hence  $m = n$ , and the direct implication follows.

Conversely, note that if  $\{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  is an interval representation of  $P$  satisfying that  $q_1 < \dots < q_n$ , then for every  $m \in [n - 1]$ ,

$$\alpha(m) = |\Lambda_m| - |V_m| \leq |\Lambda_{m+1}| - |V_{m+1}| = \alpha(m+1),$$

which means that the labeling of  $P$  is canonical.  $\square$

If  $P$  is a canonically  $n$ -labeled unit interval order, and  $\mathcal{I} = \{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  is an interval representation of  $P$  satisfying  $q_1 < \dots < q_n$ , then we say that  $\mathcal{I}$  is a *canonical* interval representation of  $P$ .



Note that the image (as a multiset) of the altitude map does not depend on the labels but only on the isomorphism class of a poset. On the other hand, the altitude map  $\alpha_P$  of a canonically  $n$ -labeled unit interval order  $P$  satisfies  $\alpha_P(1) \leq \dots \leq \alpha_P(n)$ . Thus, if  $Q$  is a canonically  $n$ -labeled unit interval order isomorphic to  $P$ , then

$$(3.1) \quad (\alpha_P(1), \dots, \alpha_P(n)) = (\alpha_Q(1), \dots, \alpha_Q(n)),$$

where  $\alpha_Q$  is the altitude map of  $Q$ . Let  $A_P$  and  $A_Q$  be the antiadjacency matrices of  $P$  and  $Q$ , respectively. As  $\alpha_P(1) = \alpha_Q(1)$ , the first rows of  $A_P$  and  $A_Q$  are equal. Since the number of zeros in the  $i$ -th column (resp.,  $i$ -th row) of  $A_P$  is precisely  $|V_i(P) - 1|$  (resp.,  $|\Lambda_i(P)| - 1$ ), and similar statement holds for  $Q$ , the next lemma follows by using (3.1) and induction on the row index of  $A_P$  and  $A_Q$ .

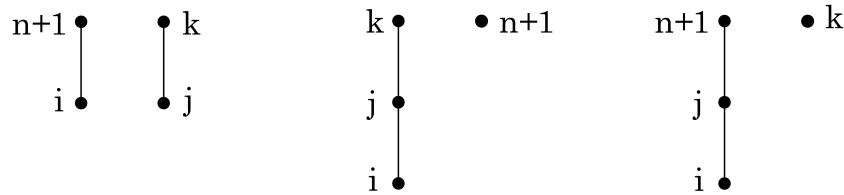
**Lemma 3.4.** *If two canonically labeled unit interval orders are isomorphic, then they have the same antiadjacency matrix.*

Now we can define a map  $\varphi: \mathcal{U}_n \rightarrow \mathcal{D}_n$ , by assigning to each unit interval order its antiadjacency matrix with respect to any of its canonical labelings. By Lemma 3.4, this map is well defined.

**Theorem 3.5.** *For each natural  $n$ , the map  $\varphi: \mathcal{U}_n \rightarrow \mathcal{D}_n$  is a bijection.*

*Proof.* Since  $|\mathcal{U}_n| = |\mathcal{D}_n| = \frac{1}{n+1} \binom{2n}{n}$ , it suffices to argue that  $\varphi$  is surjective. We proceed by induction on  $n$ . The case  $n = 1$  is immediate as  $|\mathcal{U}_1| = |\mathcal{D}_1| = 1$ . Suppose that surjectivity holds for every  $k \leq n$  and, to check that  $\varphi: \mathcal{U}_{n+1} \rightarrow \mathcal{D}_{n+1}$  is surjective, take  $D = (d_{i,j}) \in \mathcal{D}_{n+1}$ . Let  $D'$  be the submatrix of  $D$  consisting of the first  $n$  columns and the first  $n$  rows. It is not hard to see that  $D'$  is an  $n \times n$  Dyck matrix. By induction hypothesis there exists a canonically  $n$ -labeled unit interval order  $P'$  whose antiadjacency matrix is  $D'$ . Define  $P$  to be the  $(n+1)$ -labeled poset obtained by adding an element labeled by  $n+1$  to  $P'$  with exactly the following order relations:  $i <_P n+1$  if and only if either  $i = n+1$  or  $d_{i,n+1} = 0$ . Observe that  $n+1$  is a maximal element in  $P$  and that the antiadjacency matrix of  $P$  is precisely  $D$ .

We are done once we check that  $P$  is a canonically labeled unit interval order. Since  $D$  is a Dyck matrix, it follows that  $\alpha_P(1) \leq \dots \leq \alpha_P(n+1)$ . Therefore  $\alpha_P(i) < \alpha_P(j)$  implies  $i < j$  as integers, which means that the labeling of  $P$  is canonical. Finally, let us show that  $P$  is, indeed, a unit interval order. Because  $P'$  happens to be a unit interval order, it suffices to check that for any  $i, j, k \in [n]$  none of the posets



is an induced subposet of  $P$ . The first and the second subposets in the above figure cannot be induced because  $j <_P n+1$  for every non-maximal element  $j$  of  $P'$ . Consider the third subposet shown above; call it  $Q$ . If  $k <_P n+1$ , then  $Q$  cannot be induced. Suppose then that  $k$  is not comparable with  $n+1$  in  $P$ . In this case,  $k$  is maximal in  $P$ . The fact that  $j$  is not maximal in  $P$  implies that  $j < k$  as integers. On the other hand, since the labeling of  $P$  is canonical,  $i <_P j$  implies that  $i < j < k$  as integers. As  $i <_P j$ , one has that  $d_{i,j} = 0$  and so  $d_{i,k} = 0$ . Thus,  $i <_P k$ , which implies that  $Q$  is not an induced subposet of  $P$ . Hence  $P$  is a canonically  $(n+1)$ -labeled unit interval order whose antiadjacency matrix is  $D$ , and it follows by induction that  $\varphi$  is surjective, concluding the proof.  $\square$

#### 4. DESCRIPTION OF UNIT INTERVAL POSITROIDS

We proceed to describe the decorated permutation associated to a unit interval positroid. Throughout this section  $A$  is an  $n \times n$  Dyck matrix,  $B = (b_{i,j}) = \phi(A)$  is as in Lemma 2.5, and  $P$  is the unit interval positroid represented by  $B$ . Furthermore, we shall tacitly assume that the  $(2n+1)$ -th column of  $B$  is just the first column.

**Lemma 4.1.** *The following statements hold.*

- (1) *For  $1 < i \leq 2n$ , the  $i$ -th entry of the Grassmann necklace associated to the unit interval positroid  $P$  cannot contain  $i-1$ .*
- (2) *The decorated permutation associated to the unit interval positroid  $P$  does not fix any element.*

*Proof.* First, let us verify that every matrix resulting from removing one column from  $B$  still has rank  $n$ . For each  $i \in [n]$ , the submatrix of  $B$  resulting from removing the  $i$ -th column still has full rank; this is because the columns of  $B$  indexed by the set  $([n] \setminus \{i\}) \cup \{n+1\}$  are linearly independent. In addition, if  $i \in \{n+1, \dots, 2n\}$ , then the submatrix resulting from  $B$  by removing the column  $i$  also has full rank as it contains the identity matrix of order  $n$ .

Let  $\mathcal{I}_P$  be the Grassmann necklace of  $P$ . Since the matrix obtained from  $B$  by removing its  $(i-1)$ -th column has rank  $n$  and, it contains  $n$  linearly independent columns. Therefore statement (1) follows straightforwardly from the  $<_i$ -minimality of the  $i$ -th entry of  $\mathcal{I}_P$ .

Let  $\pi$  be the decorated permutation of  $P$ . Note that  $\pi$  fixes an element  $i \in [2n]$  if and only if the  $i$ -th and  $(i+1)$ -th entries of the Grassmann necklace  $\mathcal{I}_P$  are equal, which happens if and only if the matrix obtained by removing the  $i$ -th column from  $B$  has rank  $n-1$ . Hence  $\pi$  cannot fix any element.  $\square$

For the remaining of this section let  $B_j$  denote the  $j$ -th column of  $B$ . The set of principal indices of  $B$  is the subset of  $\{n+1, \dots, 2n\}$  defined by

$$J = \{j \in \{n+1, \dots, 2n\} \mid B_j \neq B_{j-1}\}.$$

We associate to  $B$  the *weight* map  $\omega: [2n] \rightarrow [n]$  defined by  $\omega(j) = \max\{i \mid b_{i,j} \neq 0\}$ ; more explicitly, we obtain that

$$\omega(j) = \begin{cases} j & \text{if } j \in \{1, \dots, n\} \\ |b_{1,j}| + \dots + |b_{n,j}| & \text{if } j \in \{n+1, \dots, 2n\}. \end{cases}$$

Since the last row of the antiadjacency matrix  $A$  has all its entries equal to 1, the map  $\omega$  is well defined. If  $j \in \{n+1, \dots, 2n\}$ , then  $\omega(j)$  is the number of nonzero entries in the column  $B_j$ . Now we find an explicit expression for the function representing the inverse of the decorated permutation associated to  $P$ .

**Proposition 4.2.** *Let  $\pi^{-1}$  be the decorated permutation associated to the unit interval positroid  $P$ . Then*

$$\pi(i) = \begin{cases} i+1 & \text{if } n < i < 2n \text{ and } i+1 \notin J \\ \omega(i) & \text{if } n < i \text{ and either } i = 2n \text{ or } i+1 \in J \\ n+1 & \text{if } i = 1 \\ i-1 & \text{if } 1 < i \leq n \text{ and } \omega(j) \neq i-1 \text{ for all } j \in J \\ j & \text{if } 1 < i \leq n \text{ and } \{j\} = J \cap \omega^{-1}(i-1). \end{cases}$$

*Proof.* Let  $\mathcal{I}_P$  be the Grassmann necklace associated to  $P$ . By Lemma 4.1(2),  $i$  cannot be in the  $(i+1)$ -th entry of  $\mathcal{I}_P$ ; we use this fact throughout this proof without further explanation. Take  $i \in \{n+1, \dots, 2n-1\}$  such that  $i+1 \notin J$ . In this case,  $B_i = B_{i+1}$ . As  $\{B_i, B_{i+1}\}$  is linearly dependent,  $i+1$  does not show in the  $i$ -th entry of  $\mathcal{I}_P$ . Therefore, the  $(i+1)$ -th coordinate of  $\mathcal{I}_P$  is obtained from its  $i$ -th coordinate by replacing  $i$  with  $i+1$ . Hence,  $\pi(i) = i+1$ .

Assume now that  $i \in \{n+1, \dots, 2n\}$  such that  $i = 2n$  or  $i+1 \in J$ . Suppose first that  $i < 2n$ . Then  $B_{i+1}$  results from replacing  $m$  ( $m > 0$ ) of the last nonzero entries of  $B_i$  by zeros. Since  $i+1 \in J$ , it follows that  $\{B_i, B_{i+1}\}$  is linearly independent and, thus, the indices  $i$  and  $i+1$  are both in the  $i$ -th coordinate of  $\mathcal{I}_P$ . Also, since the columns  $B_i, B_{i+1}, B_{\omega(i+1)+1}, \dots, B_{\omega(i)}$  are linearly dependent, not all the indices  $\omega(i+1)+1, \dots, \omega(i)$  can be in the  $i$ -th entry of  $\mathcal{I}_P$ . On the other hand, at most one index in  $\omega(i+1)+1, \dots, \omega(i)$  is missing from the  $i$ -th coordinate of  $\mathcal{I}_P$ ; this is because the submatrix of  $B$  determined by the row-index set  $\{\omega(i+1)+1, \dots, \omega(i)\}$  and the column-index set  $\{n+1, \dots, 2n\}$  has rank 1. By the minimality of the  $i$ -th entry of  $\mathcal{I}_P$  with respect to the  $i$ -order, the index of  $\{\omega(i+1)+1, \dots, \omega(i)\}$  missing in the  $i$ -th entry of  $\mathcal{I}_P$  is  $\omega(i)$ . As a result, the  $(i+1)$ -th coordinate of  $\mathcal{I}_P$  is obtained by replacing  $i$  with  $\omega(i)$ ; this is the only replacement of  $i$  that will avoid having the  $\omega(i)$ -th row of the maximal submatrix given by the columns with indices indicated by the  $(i+1)$ -th entry of  $\mathcal{I}_P$  full of zero entries. Hence  $\pi(i) = \omega(i)$ . This argument also applies when  $i = 2n$  provided that we extend the domain of  $\omega$  to  $[2n+1]$  and define  $\omega(2n+1) = 0$ .

To check that  $\pi(1) = n+1$ , it suffices to observe that the columns  $B_2, \dots, B_{n+1}$  are linearly independent and, therefore, the second entry of  $\mathcal{I}_P$  must be  $\{2, \dots, n+1\}$ .

Let us suppose now that  $1 < i \leq n$  such that  $\omega(j) \neq i - 1$  for every  $j \in J$ . The minimality of the entries of  $\mathcal{I}_P$  implies that all the indices  $i, \dots, n$  show in the  $i$ -th entry. Besides, by Lemma 4.1(1) one finds that  $i - 1$  is not in the  $i$ -th entry of  $\mathcal{I}_P$ . Since no  $j \in J$  has weight  $i - 1$ , the  $(i - 1)$ -th and  $i$ -th rows of the maximal submatrix of  $B$  determined by the column index set  $\{n + 1, \dots, 2n\}$  are equal. Consequently, the  $(i + 1)$ -th entry of  $\mathcal{I}_P$  is obtained from its  $i$ -th entry by replacing  $i$  with  $i - 1$ ; otherwise the associated maximal submatrix of  $B$  determined by the indices of the  $i$ -th entry of  $\mathcal{I}_P$  would have the  $i$ -th and  $(i + 1)$ -th rows identical. Hence  $\pi(i) = i - 1$ .

Finally, suppose that  $1 < i \leq n$  such that  $J \cap \omega^{-1}(i - 1) = \{j\}$  for some  $j$  (note that  $|J \cap \omega^{-1}(k)| \leq 1$  for every  $k \in [n]$ ). Once again the minimality of the entries of  $\mathcal{I}_P$  imply that all the indices  $i, \dots, n + 1$  show in the  $i$ -th entry (because  $i > 1$ ). Each column  $B_k$ , for  $n < k \leq 2n$  such that  $\omega(k) = i - 1$ , is a linear combination of the columns  $B_i, \dots, B_{n+1}$ . Therefore such indices  $k$  do not show in the  $i$ -th entry of  $\mathcal{I}_P$ . By Lemma 4.1(1), it follows that  $i - 1$  is also missing from the  $i$ -th entry of  $\mathcal{I}_P$ . Thus, the  $(i + 1)$ -th entry of  $\mathcal{I}_P$  is the result of replacing  $i$  with  $j \in [2n]$  satisfying that  $\omega(j) = i - 1$ ; otherwise the  $(i - 1)$ -th row of the submatrix of  $B$  determined by the indices of the  $(i + 1)$ -th entry of  $\mathcal{I}_P$  would be full of zeros. By minimality of the  $(i + 1)$ -th entry of  $\mathcal{I}_P$ , it follows that  $\pi(i) = j$ .  $\square$

We have seen that the decorated permutations of unit interval positroids do not fix any points. Indeed, as the next theorem indicates, such decorated permutations are  $2n$ -cycles satisfying certain special properties.

**Theorem 4.3.** *Decorated permutations associated to unit interval positroids on  $[2n]$  are  $2n$ -cycles  $(1 \ j_1 \ \dots \ j_{2n-1})$  satisfying the following two conditions:*

- (1) *in the sequence  $(1, j_1, \dots, j_{2n-1})$  the elements  $1, \dots, n$  appear in increasing order while the elements  $n + 1, \dots, 2n$  appear in decreasing order;*
- (2) *for every  $1 \leq k \leq 2n - 1$ , the set  $\{1, j_1, \dots, j_k\}$  contains at least as many elements of the set  $\{1, \dots, n\}$  as elements of the set  $\{n + 1, \dots, 2n\}$ .*

*Proof.* Let  $\pi^{-1}$  be the decorated permutation associated to the positroid  $P$ . From Proposition 4.2 we immediately deduce that if  $\pi(i) = j$  for  $1 < i \leq 2n$ , then  $\omega(i) = \omega(j)$  when  $i > n$  and  $\omega(i) = \omega(j) + 1$  when  $i \leq n$ ; which implies, in particular, that  $\omega(i) \geq \omega(j)$ . Suppose, by way of contradiction,  $\pi^{-1}$ , and so  $\pi$ , is not a  $2n$ -cycle. Then there is a cycle  $(i_1 \ i_2 \ \dots \ i_k)$  in the canonical cycle-type decomposition of  $\pi$  that does not contain 1. Therefore

$$\omega(i_1) \geq \omega(i_2) \geq \dots \geq \omega(i_k) \geq \omega(i_1),$$

which implies that  $\omega(i_1) = \omega(i_2) = \dots = \omega(i_k)$ . Since  $\{i_1, \dots, i_k\}$  does not contain 1, it follows that  $\{i_1, \dots, i_k\} \subseteq \{n + 1, \dots, 2n\}$ . But this is a contradiction because for all  $i, j \in \{n + 1, \dots, 2n\}$  such that  $\pi(i) = j$  we have that  $j = i + 1 > i$ . Hence the canonical cycle-type decomposition of  $\pi^{-1}$  contains only one cycle, which must have length  $2n$ .

Since  $\pi(1) = n + 1$ , it follows that  $\pi = (1 \ n+1 \ i_1 \ i_2 \ \dots \ i_{2n-2})$ , where  $\{i_1, \dots, i_{2n-2}\}$  is precisely the set  $[2n] \setminus \{1, n + 1\}$ . As

$$\omega(i_1) \geq \omega(i_2) \geq \dots \geq \omega(i_{2n-2}),$$

and  $\omega(i) = i$  for every  $i \in [n]$ , we find that the elements of the set  $\{2, \dots, n\}$  show in the cycle  $(1 \ n+1 \ i_1 \ i_2 \ \dots \ i_{2n-2})$  in decreasing order. On the other hand, Proposition 4.2 implies that the indices of equal columns of  $B$  (but perhaps the first one) show in increasing order and consecutively in the sequence  $(1, n + 1, i_1, i_2, \dots, i_{2n-2})$ . Now the fact that the weight map  $\omega$  is strictly decreasing when restricted to  $J$  implies that the elements of the set  $\{n + 1, \dots, 2n\}$  must show in increasing order in the cycle  $(1 \ n+1 \ i_1 \ i_2 \ \dots \ i_{2n-2})$ . Thus, the decorated permutation  $\pi^{-1}$  is a cycle of length  $2n$  such that when fixing 1 as the first element of the cycle the elements of the set  $[n]$  show in increasing order while the elements of the set  $\{n + 1, \dots, 2n\}$  show in decreasing order, which is condition (1).

To show condition (2), write  $\pi = (n+1 \ i_1 \ i_2 \ \dots \ i_{2n-2} \ 1)$  and suppose, by way of contradiction, that there exists  $m \in \{1, \dots, 2n - 2\}$  such that

$$(4.1) \quad |\{1 \leq j \leq m \mid i_j \in \{2, \dots, n\}\}| - 1 > |\{1 \leq j \leq m \mid i_j \in \{n + 1, \dots, 2n\}\}|,$$

and assume that such index  $m$  is minimal. By the minimality of  $m$ , one obtains that  $i_m \in \{2, \dots, n\}$ . Let  $k$  be the maximum index such that  $m \leq k$  and  $i_j \in \{2, \dots, n\}$  for each  $j = m, \dots, k$ . Note that  $k < 2n - 2$  and  $\pi(i_k) \in \{n + 2, \dots, 2n\}$ . Since

$$|\{1 \leq j \leq k \mid i_j \in \{2, \dots, n\}\}| = |\{i_k, \dots, n\}|$$

and

$$|\{1 \leq j \leq k \mid i_j \in \{n + 2, \dots, 2n\}\}| = |\{n + 2, \dots, \pi(i_k) - 1\}|,$$

it follows by (4.1) that  $(n - i_k + 1) - 1 > (\pi(i_k) - 1) - (n + 2) + 1 = \pi(i_k) - n - 2$ , which implies that  $2n - \pi(i_k) + 1 > i_k - 1$ . On the other hand, the fact that all the entries of  $A$  below and on the main diagonal equal 1 implies that  $\omega(j) \geq 2n - j + 1$  for every  $n + 1 \leq j \leq 2n$ . Since  $1 < i_k \leq n$ , one finds that  $i_k = \omega(i_k) = \omega(\pi(i_k)) + 1$ . As  $n + 1 \leq \pi(i_k) \leq 2n$ ,

$$i_k - 1 = \omega(\pi(i_k)) \geq 2n - \pi(i_k) + 1 > i_k - 1,$$

which is a contradiction. Hence, writing the decorated permutation as a  $2n$ -cycle, namely,  $\pi^{-1} = (1 \ j_1 \ \dots \ j_{2n-1})$ , we obtain that for each  $k = 1, \dots, 2n - 1$ , the set  $\{1, j_1, \dots, j_k\}$  contains at least as many elements of the set  $[n]$  as elements of the set  $\{n + 1, \dots, 2n\}$ , which is condition (2).  $\square$

## 5. A DIRECT WAY TO READ THE UNIT INTERVAL POSITROID

For the rest of the paper, let  $P$  be a canonically  $n$ -labeled unit interval order with antiadjacency matrix  $A$ . Also, let  $\mathcal{I} = \{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$  be a canonical interval representation of  $P$  (i.e.,  $q_1 < \dots < q_n$ ); Proposition 3.3 ensures the existence of such an interval representation. In this section we describe a way to obtain the decorated

permutation associated to the unit interval positroid induced by  $P$  directly from either  $A$  or  $\mathcal{I}$ . Such a description will reveal that the function  $\rho \circ \phi \circ \varphi: \mathcal{U}_n \rightarrow \mathcal{P}_n$  is a bijection.

The north and east borders of the Young diagram formed by the nonzero entries of  $A$  give us a path of length  $2n$  that we call the *semiorder path* of  $A$ . Example 5.2 sheds light upon the statement of the following theorem, which describes a way to find the decorated permutation of the unit interval positroid induced by  $P$  directly from  $A$ .

**Theorem 5.1.** *If we number the  $n$  vertical steps of the semiorder path of  $A$  from bottom to top in increasing order with  $\{1, \dots, n\}$  and the  $n$  horizontal steps from left to right in increasing order with  $\{n+1, \dots, 2n\}$ , then we get the decorated permutation of the unit interval positroid induced by  $P$  by reading the semiorder path in northwest direction.*

*Proof.* Let  $\pi^{-1}$  be the decorated permutation of the unit interval positroid induced by  $P$ , and let  $B = (I_n | A') = \phi(A)$ , where  $A' = (a'_{ij})$  and  $\phi$  is the map introduced in Lemma 2.5. Let us call *inverted path* of  $A$  the path consisting of the south and east borders of the Young diagram formed by the nonzero entries of  $A'$ . We label the  $n$  vertical steps of the inverted path of  $P$  from top to bottom in increasing order using the label set  $[n]$ , and we label the  $n$  horizontal steps from left to right in increasing order using the label set  $\{n+1, \dots, 2n\}$  (see Example 5.2). Proving the proposition amounts to showing that we can obtain  $\pi$  (the inverse of the decorated permutation) by reading the inverted path in the northeast direction.

Let  $(s_1, s_2, \dots, s_{2n})$  be the finite sequence obtained by reading the inverted path in northeast direction. Since  $a'_{n1} \neq 0$  and  $a'_{1n} \neq 0$ , the first step of the inverted path is horizontal and the last step of the inverted path is vertical. Therefore  $s_1 = n+1$  and  $s_{2n} = 1$ . Thus, to show that the permutation  $(s_1 \ s_2 \ \dots \ s_{2n})$  equals  $\pi$ , it suffices to check that  $\pi(s_k) = s_{k+1}$  for every  $k \in \{1, \dots, 2n-1\}$ .

Suppose first that the  $k$ -th step of the inverted path is horizontal. Since the  $k$ -th step of the inverted path is horizontal, then it is located right below the last nonzero entry of the  $s_k$ -th column of  $B$ . If the  $(k+1)$ -th step of the inverted path is also horizontal, then  $s_{k+1} = s_k + 1$ . These two steps being horizontal means that  $\pi(s_k) = s_k + 1$ , and so  $\pi(s_k) = s_{k+1}$ . On the other hand, if the  $(k+1)$ -th step is vertical, then  $s_k = 2n$  or  $s_k + 1$  is in the set of principal indices  $J$  of  $B$ ; in both cases,  $\pi(s_k) = \omega(s_k)$ , the number of vertical steps from the top to  $s_k$ , namely,  $s_{k+1}$ . Hence one finds again that  $\pi(s_k) = s_{k+1}$ .

Let us suppose now that the  $k$ -th step of the inverted path is vertical. This implies that  $1 \leq s_k \leq n$ . If the  $(k+1)$ -th step is also vertical, then  $s_{k+1} = s_k - 1$ . The fact that steps  $k$  and  $k+1$  are both vertical implies that  $A'$  does not contain any column with weight  $s_k - 1$ . As a result,  $\pi(s_k) = s_k - 1 = s_{k+1}$ . Finally, suppose that the  $(k+1)$ -th step is horizontal. This means that  $\{s_{k+1}\} = J \cap \omega^{-1}(s_k - 1)$  and, by Proposition 4.2, we find that  $\pi(s_k) = s_{k+1}$ .  $\square$

**Example 5.2.** The figure below displays the antiadjacency matrix  $A$  of the canonically 5-labeled unit interval order  $P_5$  introduced in Figure 2 and the matrix  $\phi(A)$  both showing their respective semiorder and inverted path encoding the decorated permutation  $\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6)$  of the positroid induced by  $P_5$ .

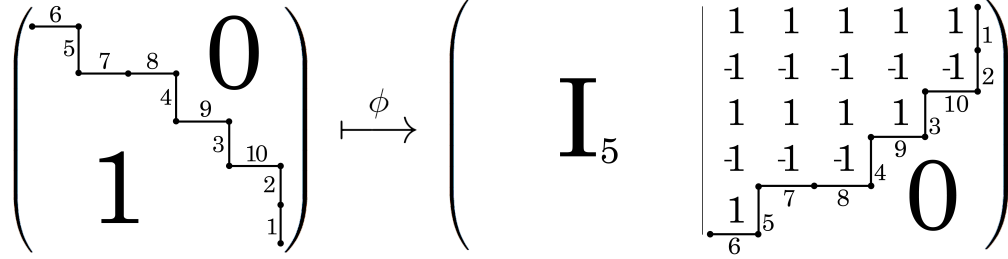


FIGURE 4. Dyck matrix  $A$  and its image  $\phi(A)$  exhibiting the decorated permutation  $\pi$  along their semiorder path and inverted path, respectively.

As a consequence of Theorem 5.1, we can deduce that the map  $\rho \circ \phi \circ \varphi: \mathcal{U}_n \rightarrow \mathcal{P}_n$ , where  $\rho$ ,  $\phi$ , and  $\varphi$  are as defined in Section 2 and Section 3, is indeed a bijection.

**Lemma 5.3.** *The set of  $2n$ -cycles  $(1\ j_1\ \dots\ j_{2n-1})$  satisfying conditions (1) and (2) of Theorem 4.3 is in bijection with the set of Dyck paths of length  $2n$ .*

*Proof.* We can assign a Dyck path  $D$  of length  $2n$  to the  $2n$ -cycle  $(1=j_0\ j_1\ \dots\ j_{2n-1})$  by thinking of the entries  $j_i \in \{1, \dots, n\}$  as ascending steps of  $D$  and the entries  $j_i \in \{n+1, \dots, 2n\}$  as descending steps of  $D$ . The fact that such an assignment yields the desired bijection is straightforward.  $\square$

**Theorem 5.4.** *The map  $\rho \circ \phi \circ \varphi: \mathcal{U}_n \rightarrow \mathcal{P}_n$  is a bijection.*

*Proof.* By definition of  $\mathcal{P}_n$ , it follows that  $\rho \circ \phi \circ \varphi$  is surjective. Since  $|\mathcal{U}_n|$  is the  $n$ -th Catalan number, it suffices to show that  $|\mathcal{P}_n| \geq \frac{1}{n+1} \binom{2n}{n}$ . To see this, take a  $2n$ -cycle  $\sigma = (1\ j_1\ \dots\ j_{2n-1})$  satisfying conditions (1) and (2) of Theorem 4.3, and consider the Dyck path  $D$  specified by  $\sigma$  as in Lemma 5.3. By Theorem 1.1, the Dyck matrix whose semiorder path is the reverse of  $D$  induce a unit interval positroid with decorated permutation  $\sigma$ . Because the decorated permutation of a positroid is unique, Lemma 5.3 guarantees that  $|\mathcal{P}_n| \geq \frac{1}{n+1} \binom{2n}{n}$ . Hence  $\rho \circ \phi \circ \varphi$  is bijective.  $\square$

**Corollary 5.5.** *The number of unit interval positroids on the ground set  $[2n]$  is the  $n$ -th Catalan number.*

We conclude this section describing how to decode the decorated permutation associated to the unit interval positroid induced by  $P$  directly from its canonical interval representation  $\mathcal{I} = \{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$ . Labeling the left and right endpoints of

the intervals  $[q_i, q_i + 1]$  by  $-$  and  $+$ , respectively, we obtain a  $2n$ -tuple consisting of pluses and minuses by reading from the real line the labels of the endpoints of all intervals in  $\mathcal{I}$ . On the other hand, we can have another *plus-minus*  $2n$ -tuple if we replace horizontal and vertical steps of the semiorder path of  $A$  by  $-$  and  $+$ , respectively, and then read it in southeast direction (see Example 5 below).

**Example 5.6.** The following figure shows the antiadjacency matrix of the canonically 5-labeled unit interval order  $P_5$  showed in Figure 2 and a canonical interval representation of  $P_5$ , both encoding the plus-minus 10-tuple  $(-, +, -, -, +, -, +, -, +, +)$ , as described in the previous paragraph.

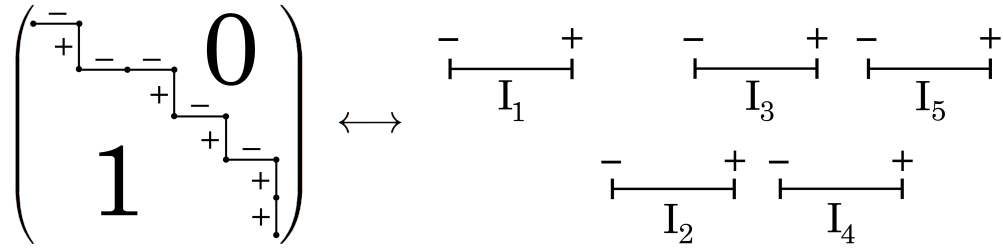


FIGURE 5. Dyck matrix and canonical interval representation of  $P_5$  encoding the 10-tuple  $(-, +, -, -, +, -, +, -, +, +)$ .

**Lemma 5.7.** *Let  $\mathbf{a}_n = (a_1, \dots, a_{2n})$  and  $\mathbf{b}_n = (b_1, \dots, b_{2n})$  be the  $2n$ -tuples with entries in  $\{+, -\}$  obtained by labeling the steps of the semiorder path of  $A$  and the endpoints of all intervals in  $\mathcal{I}$ , respectively, in the way described above. Then  $\mathbf{a}_n = \mathbf{b}_n$ .*

*Proof.* Let us proceed by induction on the order  $n$  of  $P$ . When  $n = 1$ , both  $\mathbf{a}_1$  and  $\mathbf{b}_1$  are equal to  $(-, +)$  and so  $\mathbf{a}_1 = \mathbf{b}_1$ . Suppose now that the statement of the lemma is true for every canonically  $n$ -labeled unit interval order, and assume that  $P$  is a unit interval order canonically labeled by  $[n+1]$  with antiadjacency matrix  $A$  and canonical interval representation  $\mathcal{I}$ . Let  $m$  be the cardinality of the reduced order ideal of  $n+1$  in  $P$ . By Proposition 3.2, it follows that the poset  $P \setminus \{n+1\}$  is a unit interval order canonically labeled by  $[n]$ ; therefore its associated plus-minus  $2n$ -tuples  $\mathbf{a}'_n$  and  $\mathbf{b}'_n$  are equal. Observe, in addition, that  $\mathbf{b}_{n+1}$  can be recovered from  $\mathbf{b}'_n$  by inserting the  $-$  corresponding to the left endpoint of  $q_{n+1}$  (labeled by  $2n+2$ ) in the position  $m+n+1$  (there are  $n$  left interval endpoints and  $m$  right interval endpoints to the left of  $q_{m+1}$  in  $\mathcal{I}$ ) and adding the  $+$  corresponding to the right endpoint of  $q_{n+1}$  (labeled by 1) at the end. On the other hand,  $\mathbf{a}_{n+1}$  can be recovered from  $\mathbf{a}'_n$  by inserting the  $-$  corresponding to the rightmost horizontal step of the semiorder path of  $A$  in the position  $m+n+1$  (there are  $n$  horizontal steps and  $m$  vertical steps before the last horizontal step of the semiorder path of  $A$ ) and placing the  $+$  corresponding to the vertical step labeled by 1 in the last position. Hence  $\mathbf{a}_{n+1} = \mathbf{b}_{n+1}$ , and the lemma follows by induction.  $\square$



As a consequence of Theorem 5.1 and Lemma 5.7, one obtains a way of reading the decorated permutation of the unit interval positroid induced by  $P$  directly from  $\mathcal{I}$ .

**Corollary 5.8.** *After labeling the left endpoint of  $[q_i, q_i + 1]$  by  $n + i$  and the right endpoint of  $[q_i, q_i + 1]$  by  $n + 1 - i$ , we can obtain the decorated permutation of the unit interval positroid induced by  $P$  by reading the label set  $\{1, \dots, 2n\}$  from the real line from right to left.*

*Proof.* By Lemma 5.7, the  $2n$ -tuple resulting from reading the set  $\{1, \dots, 2n\}$  as indicated in Corollary 5.8 equals the  $2n$ -tuple resulting from reading the same set from the semiorder path of  $A$  in northwest direction, as described in Theorem 5.1. Hence the corollary follows immediately from Theorem 5.1.  $\square$

**Example 5.9.** Figure 6 illustrates how to label the endpoints of a canonical interval representation of the 6-labeled unit interval order  $P$  shown in Figure 1 to obtain the decorated permutation  $\pi = (1 \ 12 \ 2 \ 3 \ 11 \ 10 \ 4 \ 5 \ 9 \ 6 \ 8 \ 7)$  of the positroid induced by  $P$  by reading such labels from the real line (from right to left).

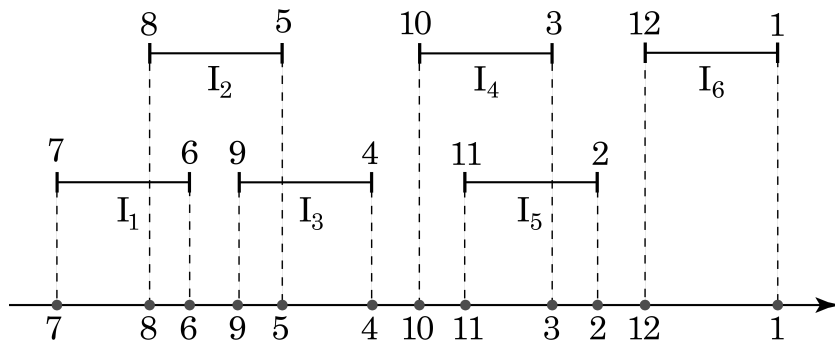


FIGURE 6. Decorated permutation  $\pi$  encoded in a canonical interval representation of  $P$ .

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